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SUBCRITICAL GROWTH OF CRACKS

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A model of stable, subcritical growth of through-cracks in thin plates under a monotone as well as a cyclic loading are considered. Viscoelastic properties of the material are taken into account, and the plastic region, in the Dugdale formulation, is assumed small.

1. Monotone loading. We consider a rectilinear through-crack in a thin elastic or viscoelastic plate. We assume that the Dugdale hypothesis holds so that the plastic deformations are concentrated in an infinity thin layer projected in the direction of the crack and the length of the plastic zone R is very small compared with the length of the crack l (see Fig. 1 which depicts the neighborhood of the tip of the crack). When the external tensile laod perpendicular to the crack is increased, the length R of the plastic zone will also increase and the tip of the crack will move along the plastic zone. The external loads will always be assumed subcritical, so that the crack growth is stable.

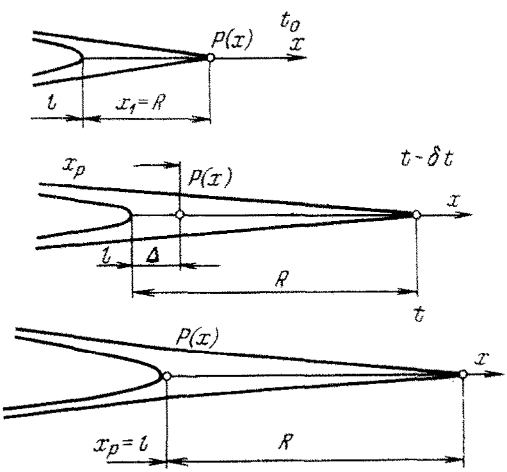

Fig. 1

First we consider the case of monotone loading. The irreversible work done at some control point P of the plastic zone in time δt preceding the fracture, is given by

$$\int_{t-\delta t}^{t} \sigma(x, \tau) U_p^{\cdot}(x, \tau)\, d\tau \tag{1.1}$$

Here the time $\tau = t$ corresponds to the instant of fracture at the point P, $\sigma(x, \tau)$ is the normal stress at the point P on the surface oriented along the x-axis which coincides with the crack, and $U_p^{\cdot}(x, \tau)$ denotes the rate of displacement of the upper edge of the palstic zone at the point P, along the normal to the x-axis. The time δt is assumed sufficiently short for the condition

$$\Delta \ll R \ll l \quad (\Delta = \delta t\, dl / dt)$$

to hold. Here Δ denotes a structural constant of the material, of the "Neuber particle" type. In physical terms, the integral (1.1) is assumed to describe the total amount of deformation at the point P over the time δt prior to failure.

Assuming that $\sigma(x, \tau)$ is constant in the zone Δ and, that the displacement over the time δt is equal to a specified critical opening of the crack U_0, we obtain from (1.1)

$$U_0 = \left(\frac{\partial U_p}{\partial x_1}\right)_{t-\delta t} \delta x_1 + \left(\frac{\partial U_p}{\partial R}\right)_{t-\delta t} \delta R + \ldots \tag{1.2}$$

Here the variables x and t are replaced by the distance from the crack tip x_1 and the size of the plastic zone R, both being unknown functions of time. Since $x = x_1 + l$ (see Fig. 1) and the value of x for the control point P is fixed with respect to time, we can replace $\partial / \partial x_1$ by $(-\partial / \partial l)$ and δx_1 by $(-\delta l)$.

For a material which is elastic outside the plastic zone, the displacement U_p is given by

$$U_p(x_1, R) = \frac{4\sigma_s R}{\pi E}\left(\sqrt{1 - \frac{x_1}{R}} - \frac{x_1}{2R} \ln \frac{1 + \sqrt{1 - x_1/R}}{1 - \sqrt{1 - x_1/R}}\right) \tag{1.3}$$

where σ_s denotes the yield point and E is Young's modulus. In this case the criterion (1.2) becomes

$$\frac{dR}{dl} = \frac{\pi E U_0}{4\sigma_s \Delta} - \frac{1}{2}\left(1 + \ln \frac{4R}{\Delta}\right) \tag{1.4}$$

Integrating this equation we obtain a curve depicting the resistance against the crack growth. Cherepanov was first to construct such a curve analytically, using his energetic theory developed in [1]. In order to assess the effect of growth predicted by equation (1.4), it is sufficient to compare the size of the plastic zone R_0 at the initial instant of crack formation

$$R_0 = \frac{\pi E}{4\sigma_s} U_0 \tag{1.5}$$

with the maximum value R_∞ for the size of the plastic zone corresponding to the stationary stage of the crack growth when $dR / dl = 0$

$$R_\infty = \frac{\Delta}{4} \exp\left(\frac{\pi E U_0}{2\sigma_s \Delta} - 1\right) \tag{1.6}$$

Since $U_0 \sim \Delta$, the Ratio R_∞ / R_0 is small for the typical quantity σ_s / E.

Next we assume that the material outside the plastic zone is linearly viscoelastic, i.e.

$$S_{i,j} = \int_0^t G_1(t-\tau) \frac{de_{i,j}(\tau)}{d\tau} d\tau, \quad \sigma = \int_0^t G_2(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \tag{1.7}$$

Here $S_{i,j}$ and $e_{i,j}$ are the deviators, while σ and ε are the first invariants of the stress and deformation tensors, respectively, $G_1(t)$ is the modulus of relaxation for pure shear and $G_2(t)$ is the modulus of relaxation for pure compression.

In this case the displacement U_p is given by the formula

$$U_p(x_1, R) = U_p^\circ\{x_1(t), R(t)\} + \int_{t_0}^{t} \frac{\Omega(t-\tau)}{\Omega(0)} U_p^\circ\{x_1(\tau), R(\tau)\}\, d\tau \qquad (1.8)$$

Here the lower limit t_0 coincides with the instant of time at which the control point P enters the plastic zone, the superscript ^{0} denotes the elastic component of the displacement U_p and the function $\Omega(t)$ has the form

$$\Omega(t) = \underset{S\to t}{L^{-1}} \left\{ \frac{2\,[2G_1^*(S) + G_2^*(S)]}{S^2\,[G_1^*(S) + 2G_2^*(S)\,G_1^*(S)]} \right\} \qquad (1.9)$$

where the asterisk denotes a Laplace transform and L^{-1} its inverse.

Using (1.8) we can reduce the criterion (1.2) to the form

$$\frac{dR}{dl} = \left(\frac{4E}{\pi\sigma_s}\right)\left(\frac{U_0}{\Delta}\right) - \frac{1}{2}\left(1 + \ln\frac{4R}{\Delta}\right) - R\,[\delta\Omega(t)]\,\Omega^{-1}(0) \qquad (1.10)$$

The influence of the viscoelastic deformations on the stable growth of cracks is given by the last term of (1.10), or more accurately, by the increment $\delta\Omega(t) = \Omega(\delta t) - \Omega(0)$.

Expanding the function $\Omega(\delta t)$ into a series, we obtain the relation connecting it with the rate of change of the size of the plastic zone, and the rate of loading $Q^\cdot$

$$\delta\Omega(t) = \left(\frac{dR}{dl} - \frac{\partial R}{\partial l}\right)\frac{\Omega^\cdot(0)}{Q^\cdot}\left(\frac{\partial R}{\partial R}\right)^{-1}\Delta$$

Finally, we write the equation (1.10) in the form

$$\frac{dR}{dl} = \left[\frac{4U_0E}{\pi\sigma_s} - \frac{\Delta}{2}\left(1 + \frac{4R}{\Delta}\right) + C\Delta\left(\frac{\partial R}{\partial Q}\right)^{-1} R\left(\frac{\partial R}{\partial l}\right)\right]\Delta^{-1} \times \qquad (1.11)$$
$$\left[1 + CR\left(\frac{\partial R}{\partial Q}\right)^{-1}\right]^{-1}$$

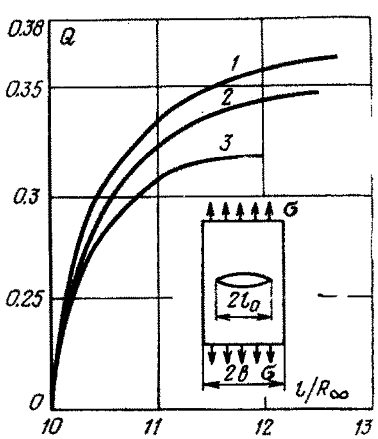

Fig. 2

The parameter C has a substantial influence on the resistance to the crack growth. Fig. 2 shows the results of numerical integration of (1.11) in terms of the dimensionless load $Q = \pi\sigma / (2\sigma_s)$ and dimensionless crack length l / R_∞ for several fixed values of the parameter $\lambda = \pi l / (2l)$ (arrows indicate the onset of instability). The curves 1 — 3 correspond to the values of $\lambda = 0.6$, 0.4 and 0.6.

For the materials insensitive to change in the rate of loading, i.e. when $C = 0$, the equation (1.11) becomes (1.4). We note that the curve depicting the resistance to the crack growth becomes the universal curve, i.e. the form of the curve depends neither on the geometry of the sample, nor on the initial length of the crack, only when $C = 0$. The existence of such a universal curve was predicted by Rice [2] for the longitudinal shear cracks, and by Cherepanov for the normal fracture cracks [1]. Clearly, the universality is violated as soon as the material becomes responsive to the loading rate.

When the loading, the form of the sample and the location of the crack are all given, the relationship connecting R with l and Q can be assumed known. It follows that the expression (1.11) represents a first order ordinary differential equation connecting the crack length l with the dimensionless load Q.

2. **Cyclic loading.** We consider the growth of fatigue cracks as a sequence of slow growth intervals each of which is described by the equation (1.11). Integrating this equation over a single loading cycle, we obtain the incremental crack growth over a single cycle

$$dl = \int_{R_{\min}}^{R_{\max}} F\,dR + \oint CR\left(\frac{dR}{\partial Q}\right)^{-1} F\,dR \tag{2.1}$$

$$F = \{{}^1\!/_2 \ln (R_\infty / R) + CR\,(\partial R / \partial l)\,(\partial R / \partial Q)^{-1}\}^{-1}$$

Since the crack length does not alter appreciably during a single cycle, it is assumed to be constant in (2.1). In this case the variable of integration R can be replaced by Q.

We note that the first integral in (2.1) extends over the ascending portion of the cycle only, while the second integral covers the complete cycle. This implies a different physical interpretation for each term involved. The first integral accounts for the slow growth of the crack which can only occur during the period of active loading. The second integral is governed by the viscoelastic behavior of the material and therefore some growth can occur at the sustained load or even at the decreasing load. If we therefore decompose the cycle into an ascending $(0 \leqslant t \leqslant T / 2)$ and descending $(T / 2 \leqslant t \leqslant T)$ branches, then the incremental crack growth over the above segments is given by the expressions

$$(dl)_+ = \int_{Q_{\min}}^{Q_{\max}} \left(\frac{\partial R}{\partial Q}\right) F\,dQ \tag{2.2}$$

$$(dl)_- = \int_{T/2}^{T} \left(\frac{R}{Q^{\cdot}}\right) F\,dt$$

The integrals (2.2) can be computed for any given loading regime $Q = Q(t)$. The first integral can be evaluated in terms of the maximum and minimum load levels within the cycle without knowing the precise dependence of Q on time. The computation of the second integral in (2.2) however requires the knowledge of the function $Q(t)$.

We illustrate the application of the above relations by computing the rate of growth of an isolated fatigue crack in an infinite plate. From (2.2) we obtain the following expression for the growth of a fatigue crack:

$$(dl)_+ = l \int_{Q_{\min}}^{Q_{\max}} \left(2Q + \frac{BQ^2}{Q^\cdot}\right) G\, dQ \qquad (2.3)$$

$$G = \{\ln(2R_\infty / lQ^2) - Q^2\}^{-1}$$

$$(dl)_- = CQ^\cdot l \int_{Q_{\min}}^{Q_{\max}} \left(\frac{Q^2}{Q^\cdot}\right) G\, dQ$$

$$\frac{dl}{dN} = l \oint \left(2\alpha Q + \frac{Q^2}{Q^\cdot}\right) G\, dQ$$

Here N is the number of cycles, the coefficient α is equal to one for the ascending part of the loading cycle, and to zero for the descending part.

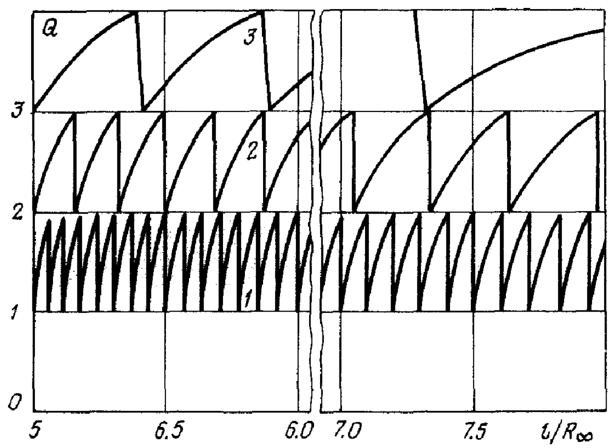

Fig. 3

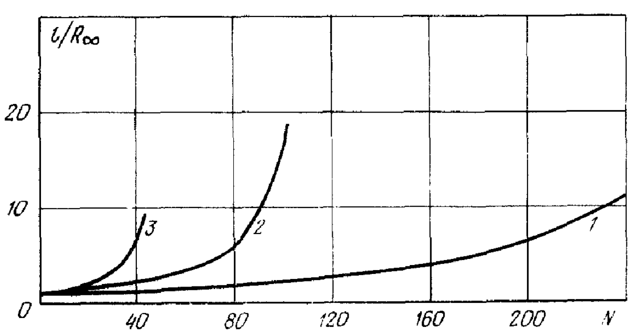

Fig. 4

Figures 3 —5 depict some of the results of computing the integral (2.3) in the case when the cyclic loading is a simple sawtooth-like function

$$Q(t)=\begin{cases}(-nT+t)\,Q^{\cdot}, & nT\leqslant t\leqslant(2n+1)\,T/2\\ (nT-t)\,Q^{\cdot}, & (2n+1)\,T/2\leqslant t\leqslant T\,(n+1)\end{cases}$$

In Fig. 3 the dimensionless crack length $l\,/\,R_{\infty}$ serves as the abscissa and the dimensionless load Q as the ordinate. The numerical integration was carried out using the Runge —Kutta subroutine. The loading frequency chosen was $\omega = 5\sec.^{-1}$ ($Q^{\cdot} = \pi^{-1}\sec.^{-1}$), $B = 0.1$ (here $B = CQ^{\cdot}$), $Q_{\min} = 0.1$, 0.2 and 0.3, $Q_{\max} = 0.2$, 0.3 and 0.4 (the corresponding curves are 1 —3). As we see, the growth of a fatigue crack is strongly asymmetric already within a singe cycle.

The contribution of the descending branch becomes more pronounced at higher loading levels and larger values of the parameter C.

Figure 4 depicts the growth of a fatigue crack as a function of the number of cycles N. Fig. 5 shows the dependence of the rate of crack growth on the amplitude of the stress intensity coefficient $\Delta K = K_{\max} - K_{\min}$(the quantity $\ln\,(\Delta K\,/\,K_{\infty}$ is plotted along the abscissa). The loading conditions in the figures 3 —5 are identical. The last graph uses logarithmic coordinates. We note that the graphs in Fig. 5 are nearly linear over a certain range. The slope of this straight line segments is nearly equal to 4.0, in accordance with the Paris law.

We check the Miner's law of cumulative damage by constructing two programs of investigation under a variable load. The results are given in the table and completely confirm the Miner's law. The results obtained under the extremal conditions of loading are considered separately.

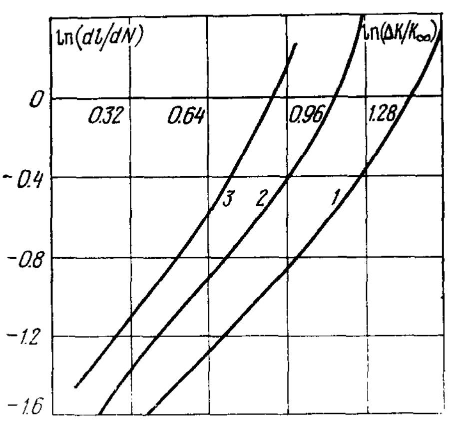

Fig. 5

Let a cyclic loading be applied first with $N_i\,/\,N_f = 150/202$ (N_f denotes the life of the sample, i.e. the number of cycles prior to failure, and N_i is the current number of cycles), at the lowest level of loading ($0.1 \leqslant Q \leqslant 0.2$ and $B = 1$). Next, a half of the cycle is applied at the highest level of loading ($0.3 \leqslant Q \leqslant 0.4$). This gives $\Sigma N_i\,/\,N_f = 0.743$. Thus the deviation from the Miner's law is this case about 26%.

Let now a cyclic loading be applied with $N_i\,/\,N_f = 22\,/\,22$, at the highest level of loading ($0.3 \leqslant Q \leqslant 0.4$ and $B = 1$). This is followed by 52 additional cycles at the lowest loading level ($0.1 \leqslant Q \leqslant 0.2$). Then $\Sigma N_i\,/\,N_f = 1.26$ (deviation from the Miner's law is again 26%). The above cases produce results with relatively large deviations from the Miner's law. Nevertheless, bearing in mind the fact that the above results are highly specific, in should not be assumed that $\Sigma N_i\,/\,N_f$ differs much from unity.

Table 1

Q	B	N_f	N_i	$\Sigma N_i / N_f$	N_i	$\Sigma N_i / N_f$
	0.01	48	12	1.0267	28	0.988
0.3—0.4	0.1	43	11		25	
	1.0	22	6		13	
	0.01	110	20	1.0282	19	0.981
0.2—0.3	0.1	102	18		17	
	1.0	59	11		10	
	0.01	311	120	1.0285	72	1.003
0.1—0.2	0.1	292	113		68	
	1.0	202	77		49	

The results obtained represent a further development of [3 —6]. Recently, the author obtained (*) an equation resembling formally (1.4) but valid also in the case when the size of the plastic zone is arbitrary. In the present case the equation differs from (1.4) only by the presence of a factor $\tilde{R}/I$ in the right hand side. The equation can be integrated in closed form and gives

$$R = R_0\left(\frac{l}{l_0}\right)^{\lambda_0+1}\exp\left\{-\frac{1}{4}\ln^2\frac{l}{l_0}\right\}$$

$$\lambda_0 = \frac{u_0}{\Delta}\left(\frac{\pi E}{4\sigma_s}\right) - \frac{3}{2} - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\frac{l_0}{\Delta}$$

i.e. λ_0 depends on the geometry and comliance of the material.

In conclusion the author thanks Prof. Juricic for writing the computational programs.

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